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# Calculation of the wetted area of a planing hull with a chine

E. M. CASLING* and G. W. KING**

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SUMMARY

The extent to which a low-aspect-ratio flat ship with a chined hull is wetted when planing at infinite Froude number is investigated. A numerical method of solution for the wetted area, which is applicable to more general planing problems, is presented. The results obtained by this method are compared with those found by solving the inverse problem of determining the hull shape which produces a given waterplane shape and are shown to be in excellent agreement. Results are also presented which indicate that a 'vertical' chine may be used to fix the shape of the wetted region.

## 1. Introduction

The motion of a boat at high speed on a free surface (planing) has been studied extensively, both experimentally and theoretically, by a large number of authors. A bibliography and brief review of some of the more important (in the authors' view) papers is given by Casling [1]. However, very few authors have been concerned with problems of indeterminacy of the shape of either the planing hull or the free surface, three exceptions being Tuck [3], Oertel [4] and Casling [1, 5].

The work presented here is an application to chined hulls of results obtained by one of the authors in a previous paper (Casling [5]). It is an extension of the low-aspect-ratio flat-ship theory of Tuck [3] for infinite Froude number and shows how the extent to which a hull is wetted is fixed completely be the physical characteristics of the hull.

The relationship is expressed as an integral equation, which must then be solved for the function describing the shape of the wetted area. Difficulties arise because the range of integration depends on the inverse of the function defining the waterplane shape and therefore must also be determined as part of the solution to the problem. A numerical method for inverting the integral equation is described and particular examples which indicate its accuracy are discussed. The method is applicable to any planing hull, not necessarily one with a chine, and can be readily generalised to solve other integral equations in which the range of integration is unknown, but not necessarily dependent on the inverse of the function being determined.

Results which indicate how a vertical chine may be used to predetermine the waterplane shape are also presented.

* School of Mathematics & Computer Studies, South Australian Institute of Technology, The Levels, South Australia, Australia

** Department of Applied Mathematics, University of Adelaide, Adelaide, South Australia, Australia

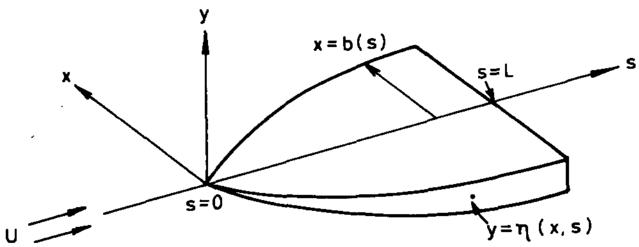


Figure 1. The coordinate system.

## 2. Mathematical formulation and solution of the general problem

The general problem of the motion of a ship, which is slender as well as flat, will be described briefly first. For a more detailed discussion of the following derivation, the reader is referred to Tuck [3] or Casling [5].

We assume that a low-aspect-ratio flat ship is moving with speed $U$ in an $(x, y, s)$ coordinate system, whose origin is fixed to the bow, with $x$ to starboard, $y$ normal to the mean plane of the hull and $s$ in the streamwise direction (see Figure 1). Assuming the flow is irrotational, the velocity field is given by

$$\mathbf{q} = \nabla\Phi = \nabla(Us + \phi),$$

where $\phi$ is the perturbation velocity potential.

The hull surface is defined by

$$y = \eta(x,s)$$

for $|x| < b(s)$ and $s < L$, where $b(s)$ is the half waterplane width and $L$ is the length of the ship. Outside the hull surface, $y = \eta(x, s)$ describes the free-surface elevation caused by the motion of the ship. We will assume that the hull shape $\eta(x, s)$ is a strictly monotone-decreasing function of $s$ and that the function describing the shape of the waterplane, $x = b(s)$, is strictly monotone-increasing. The second assumption ensures that the flow does not separate from the leading edges of the hull upstream of the transom stern. Since the flow is symmetric, only $x \geqslant 0$ will be considered.

The mathematical problem to be solved is to find the perturbation velocity potential, $\phi$, given that

$$\phi_{xx} + \phi_{yy} = 0 \tag{2.1}$$

in the region $y < 0$ subject to the conditions

$$\phi_y = U\eta_s \qquad \text{on } y = 0 \tag{2.2}$$

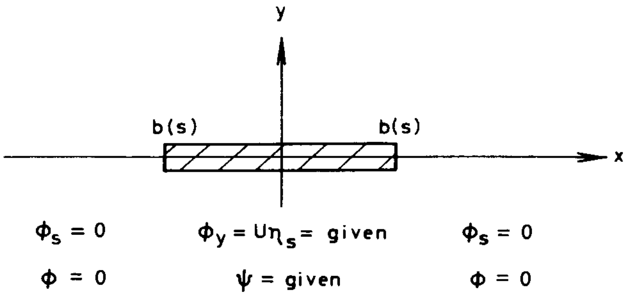


Figure 2. Cross-flow plane.

and

$$P/\rho + U\phi_s = 0 \qquad \text{on } y = 0, \tag{2.3}$$

with the appropriate radiation condition at infinity (see Figure 2). $P$ is defined as the excess of pressure over atmospheric at the free surface. Equations (2.2) and (2.3) are the linearised boundary conditions obtained from the exact hull boundary condition and kinematic free-surface condition, and the dynamic free-surface condition respectively when the small-draft and low-aspect-ratio approximations are made. These linearised boundary conditions are applied on $y = 0$ because, when the small-draft approximation is made, the hull reduces to its projection onto the plane $y = 0$. $\phi$ satisfies the two-dimensional Laplace equation (2.1) because, since the ship is slender, $\phi$ is the potential for the problem in the crossflow plane.

The solution to this problem in terms of the stream function $\psi(x,s)$ was given by Casling [5] in the form

$$\psi(x,s) = (x^2 - b^2(s))^{\frac{1}{2}} H_{b(s)}\psi(x,s)\,(b^2(s) - x^2)^{-\frac{1}{2}}, \quad x > b(s), \tag{2.4}$$

where

$$H_{a(s)}f(x,s) \overset{\text{def.}}{=} \frac{1}{\pi} \int_{-a(s)}^{a(s)} \frac{d\xi}{x - \xi} f(\xi,s)$$

is the Hilbert transform (see Tricomi [6], p. 173) of a function $f$ on the interval $(-a(s), a(s))$. When $-a(s) < x < a(s)$, the integral is treated as a Cauchy principal-value integral.

Expressions for the free-surface elevation may be derived from equation (2.4) (see Casling [5]), because equation (2.2) and the Cauchy-Riemann equation

$$\phi_y = -\psi_x,$$

give

$$\eta_s(x,s) = -\psi_x(x,s)/U.$$

Thus, when $x > b(s)$,

$$\eta(x,s) = -\int_0^s d\sigma (x^2 - b^2(\sigma))^{-\frac{1}{2}} H_{b(\sigma)} \eta_\sigma(x,\sigma) (b^2(\sigma) - x^2)^{\frac{1}{2}} \quad (2.5)$$

and, when $|x| < b(s)$,

$$\eta(x,s) = \int_0^s d\sigma\, \eta_\sigma(x,\sigma) + c(x), \quad (2.6)$$

where

$$c(x) = -\int_0^{s_0(x)} d\sigma (x^2 - b^2(\sigma))^{-\frac{1}{2}} \cdot \frac{1}{\pi} \int_{-b(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi} (\eta_\sigma(\xi,\sigma) - \eta_\sigma(x,\sigma)) (b^2(\sigma) - \xi^2)^{\frac{1}{2}}$$
$$- x \int_0^{s_0(x)} d\sigma (x^2 - b^2(\sigma))^{-\frac{1}{2}} \eta_\sigma(x,\sigma) \quad (2.7)$$

and $s = s_0(x)$ is the station at which $x = b(s)$.

Equation (2.7) expresses the relationship between the waterplane shape $b(s)$, hull slope $\eta_s(x,s)$ and the underwater hull shape described by $c(x)$ and was discussed in some depth by Casling [5]. It means that, given any two of the above three functions, the third one *cannot* be fixed a priori but must be determined from this relationship.

## 3. A planing hull with a chine

A chine is a discontinuity in $\eta_x(x, s)$ at some offset $x = B(s)$, where $B(s) < b(s)$. For two examples of such a section shape, see Figures 3 and 4. The chine may occur along either a fixed offset, say $x = B$, or an offset which is a sufficiently smooth function of the station $s$. In the latter case, we will assume that $x = B(s)$ is a strictly monotone-increasing function of $s$.

The discontinuity may arise in a number of ways. Firstly, there may be a jump in the lateral slope of the section shape, while $\eta(x, s)$ remains continuous (see Figure 3). That is,

$$\eta_x(B(s)^+, s) \neq \eta_x(B(s)^-, s),$$

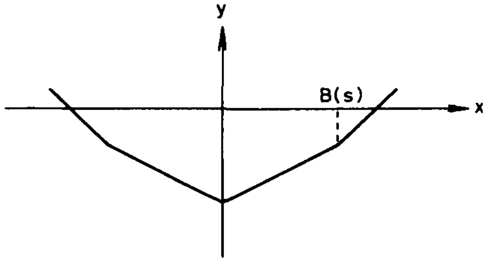

Figure 3. Cross-section of a chine.

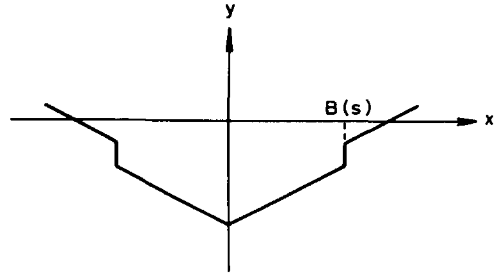

Figure 4. Cross-section of a chine.

but

$$\eta(B(s)^+,s) = \eta(B(s)^-,s), \tag{3.1}$$

for fixed station $s$. Secondly, there may be a positive jump of $O(\alpha) = O(\text{draft/length})$ in $\eta(x, s)$ at $x = B(s)$, but no change in the lateral slope (see Figure 4). That is,

$$\eta(B(s)^+,s) = \eta(B(s)^-,s) + h(s),$$

where $h(s) > 0$ and $h(s) = 0(\alpha)$, and

$$\eta_x(B(s)^+,s) = \eta_x(B(s)^-,s). \tag{3.2}$$

Of course, the combination of these two cases, in which there is a jump in both $\eta$ and $\eta_x$ at $x = B(s)$, may also occur. We will derive results for this general case. That is,

$$\eta_x(B(s)^+,s) \neq \eta_x(B(s)^-,s) \tag{3.3}$$

and

$$\eta(B(s)^+,s) = \eta(B(s)^-,s) + h(s). \tag{3.4}$$

It should be noted that any chine which produces a non-monotone waterplane shape, $x = b(s)$, is not permissible, since separation of the flow will occur from such a shape forward of the trailing edge and the problem is considerably altered. Provided $\eta_x(x, s)$ is non-negative for $x \geqslant 0$ and $0 \leqslant s \leqslant L$, difficulties do not arise. Also, it is clear that the following results may be readily generalised to the case of a finite number of discontinuities in $\eta_x(x, s)$.

As yet, we have made no assumptions concerning the behaviour of $\eta_s(x, s)$ at $x = B(s)$. Since, from equation (3.4), $\eta(x, s)$ has a step of $h(s)$ at $x = B(s)$, a continuous function $g(x, s)$ may be chosen so that

$$\eta(x,s) = g(x,s) + u(x - B(s))h(s), \quad 0 \leqslant x \leqslant b(s),$$

where $u(t)$ is the unit step function, or Heaviside function. Therefore, unless $h(s) = 0$, both $\eta_s(x, s)$ and $\eta_x(x, s)$ have a delta-function discontinuity at $x = B(s)$. For $x > B(s)$, $\eta_s(x, s) = g_s(x, s) + h'(s)$ and so, in the limit as $x \to B(s)^+$,

$$\eta_s(B(s)^+,s) = g_s(B(s)^+,s) + h'(s).$$

Thus, in general, $\eta_s(B(s)^+,s)$ will differ from $\eta_s(B(s)^-,s)$.

We assume that the chine starts at some station $s = s_c$, where $0 \leqslant s_c \leqslant L$, and that $B(s_c) = b(s_c)$ (see Figures 5 and 6). When $s \leqslant s_c$, that is, before the chine, the hull has no discontinuities and the results given in Section 2 apply directly. Therefore, when $0 \leqslant x \leqslant s_c$, the free-surface elevation is given by equations (2.5) and (2.6). When $s > s_c$, the free-surface elevation is described by the same two equations, but it must be remembered that, for a chine at varying

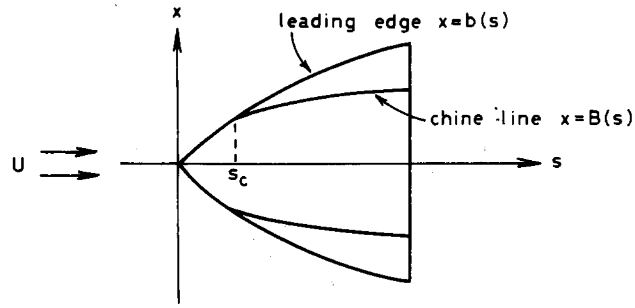


Figure 5. Chine starting at station $s = s_c$.

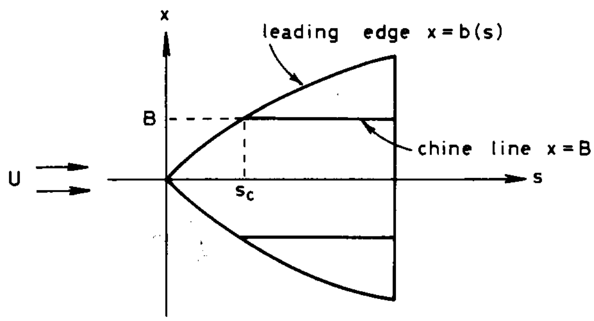


Figure 6. Chine at fixed offset $x = B$.

offset, $\eta_s$ now has a discontinuity at $x = \pm B(s)$ for $s > s_c$. So, even though there is no explicit chine dependence in the equations for the free-surface elevation, the chine does affect the flow. This effect may also be observed by considering the following. As was shown in Section 2, given $\eta_s$ and $b(s)$, the function $c(x)$ is fixed. If a chine is introduced, then either $\eta_s$ or $b(s)$ must change for equation (2.7) to be satisfied by the new $c(x)$. So the effect of the chine on the free surface is felt indirectly through $x = b(s)$.

The direct problem of determining the extent of the wetted area for a given hull shape involves solving equation (2.7) for the function $x = b(s)$. The task is more complicated for a chined hull due to the implicit presence of the starting station of the chine, $s_c$. This point, where the free surface crosses the chine line, is not known in advance, because the extent of the wetted area is also unknown. However, since $x = B(s)$ is known, $s_c$ may be determined, once $b(s)$ is known, by finding where the curves $x = B(s)$ and $x = b(s)$ intersect. If a particular waterplane shape is required, that is, if $x = b(s)$ is given, then equation (2.7) fixes the hull geometry for a given chine line, $x = B(s)$.

Casling [5] has solved analytically the direct problem for the simple case in which $\eta_s$ is independent of $x$, that is, $\eta_s = -f(s)$, although an explicit form for the function $x = b(s)$ was not always possible. In Section 4, a method for numerically solving equation (2.7) for the function defining the waterplane of an arbitrary hull shape (not necessarily chined) is discussed.

## 4. Numerical solution technique

So that the waterplane shapes of different flat ships can be calculated, a numerical technique for solving equation (2.7) with $\eta_s(x, s)$ and $c(x)$ given is needed. Equation (2.7) may be written as

$$c(b(s)) = -\int_0^s d\sigma(b^2(s) - b^2(\sigma))^{-\frac{1}{2}} \left\{ \frac{1}{\pi} \int_{-b(\sigma)}^{b(\sigma)} \frac{d\xi}{b(s) - \xi} \right.$$
$$\left. (\eta_\sigma(\xi,\sigma) - \eta_\sigma(b(s),\sigma))\,(b^2(\sigma) - \xi^2)^{\frac{1}{2}} + b(s)\eta_\sigma(b(s),\sigma) \right\}$$
$$= -\int_0^s d\sigma\, k(b(s),\sigma;b)\,(b^2(s) - b^2(\sigma))^{-\frac{1}{2}} \tag{4.1}$$

where $k(b(s),\sigma;b)$ is a function of the waterplane shape $b(s)$ and is given by the expression in braces. For the problem of a low-aspect-ratio flat ship, $b(s)$ is, by assumption, a strictly increasing function so that when equation (4.1) is evaluated at $b(s) = x$, the upper bound of the range of integration is $s = b^{-1}(x)$.

To find a numerical solution, $N + 1$ points $s_i$, such that $0 = s_0 < s_1 < \ldots < s_N$, are distributed along the s-axis with $s_N$ being the position of the last station at which we wish to find the waterplane. At each station $s_i$, the corresponding position of the waterplane $b_i = b(s_i)$ needs to be calculated ($b_0 = 0$). So that the integral in equation (4.1) can be evaluated, an interpolation process between the computed $b_i$ is needed to give us the function $b$. Equation (4.1) may be written as

$$F(b(s);b) = c(b(s)) - \int_0^s d\sigma\, k(b(s),\sigma;b)\,(b^2(s) - b^2(\sigma))^{-\frac{1}{2}} \tag{4.2}$$

where we are required to find $b_i, i = 1, 2, \ldots, N$ such that

$$F(b_i;b) = 0. \tag{4.3}$$

In order to start the process of solving equation (4.3), an initial approximation $b_1^0$ for $b_1$ is needed. Then, using a numerical solution procedure for equations in one variable, an approximation $b_1^*$ to the actual value $b_1$ is calculated. From this, by extrapolation, an initial guess $b_2^0$ for $b_2$ can be made and used to calculate $b_2^*$ by the same procedure as $b_1^*$. Thus, a numerical solution $b_i^*$, $i = 1, \ldots, N$ to (4.3) can be found, provided the integral in equation (4.2) can be evaluated.

The numerical integration for this integral needs to be done carefully, as the integrand has a square-root singularity at the upper end of the range of integration. A method for numerically evaluating such an integral can be constructed by not calculating the integrand at the point at which the singularity occurs (see Davis & Rabinowitz [2]). The accuracy of the quadrature procedure can be improved by distributing quadrature points over the interval of integration in such a way that the nature of the singularity is accounted for.

As the singularity is of the square root type, the $M + 1$ quadrature points $s^i_k$ for the integral from 0 to $s_i$ were chosen as

$$s^i_k = s_i \sin\left(\frac{\pi}{2M}(k - \tfrac{1}{2})\right) \text{ for } k = 1, \ldots, M$$

and

$$s^i_0 = 0. \tag{4.4}$$

This is equivalent to using integration by substitution and then having a regular grid of quadrature points in terms of the new variable. Trapezoidal integration was used on each of the $M$ intervals $[s^i_{k-1}, s^i_k]$, so that a composite trapezoidal rule was applied to the whole interval $[0, s^i_M]$. This method of integration was found to give satisfactory accuracy for $M \simeq 100$. The integral in equation (4.2) at $b_i$ was therefore approximated by

$$\int_0^{s_i} d\sigma\, F(\sigma) \simeq \int_0^{s^i_M} d\sigma\, F(\sigma)$$

$$\simeq \tfrac{1}{2}\left\{F(s^i_0)(s^i_k - s^i_0) + \sum_{k=1}^{M-1} F(s^i_k)(s^i_{k+1} - s^i_{k-1})\right.$$

$$\left. + F(s^i_M)(s^i_M - s^i_{M-1})\right\} \tag{4.5}$$

where

$$F(\sigma) = k(b(s_i), \sigma; b)\,(b^2(s_i) - b^2(\sigma))^{-\frac{1}{2}}.$$

For the numerical examples used here, the method of false position was used to solve equation (4.3), as this took less evaluations of $F(b_i;b)$ than comparable methods, such as Newtonian iteration. Hence the number of times the integral needed to be evaluated was decreased. In practice, it was the evaluation of the integrals which took the most time. In the computer program, linear interpolation and extrapolation were used. As a check on the accuracy of the solution, after all the $b_i$'s were calculated a cubic Hermite interpolation function was fitted to the $b_i$'s and used to recalculate equation (4.2). The resulting values of $F(b_i;b)$ give a good estimate of the errors in the numerical solution.

## 5. Numerical results

It is also possible to obtain results by solving the indirect problem analytically. In this case, we determine the hull shape which produces the required waterplane shape. If we then use the determined hull shape as input for the numerical procedure and calculate the expected wetted area, we can see how well the method works by comparing the result with the initial required waterplane shape.

For example, suppose the hull shape is given by

$$\eta(x,s) = \begin{cases} -\alpha s + c_1(x), & x \leqslant B(s), \\ -(\alpha-\gamma)s + c_2(x), & B(s) < x \leqslant b(s), \end{cases}$$

where $\alpha$ and $\gamma$ are small constants, and the chine starts at the bow and occurs along a varying offset with

$$B(s) = \theta s,$$

for small constant $\theta$.

Then, equation (2.7) becomes

$$c(x) = \gamma \int_0^{s_0(x)} d\sigma (x^2 - b^2(\sigma))^{-\frac{1}{2}} F(x,\sigma) + (\alpha-\gamma)x \int_0^{s_0(x)} d\sigma (x^2 - b^2(\sigma))^{-\frac{1}{2}} \tag{5.1}$$

where

$$c(x) = \begin{cases} c_1(x), & x \leqslant B(s), \\ c_2(x), & B(s) < x \leqslant b(s), \end{cases}$$

and

$$F(x,\sigma) = \frac{1}{\pi} \int_{-\theta\sigma}^{\theta\sigma} \frac{d\xi}{x-\xi} (b^2(\sigma) - \xi^2)^{\frac{1}{2}}$$

$$= \frac{2x}{\pi} \sin^{-1}(\theta\sigma/b(\sigma)) + \frac{(x^2 - b^2(\sigma))^{\frac{1}{2}}}{\pi} \times$$

$$\left\{ \sin^{-1}\left(\frac{b^2(\sigma) - x\theta\sigma}{b(\sigma)(x-\theta\sigma)}\right) - \sin^{-1}\left(\frac{b^2(\sigma) + x\theta\sigma}{b(\sigma)(x+\theta\sigma)}\right) \right\}.$$

If we now assume that the waterplane is triangular with

$$b(s) = \beta s,$$

for a small constant $\beta > \theta$, then equation (5.1) may be integrated analytically, the result being

$$c(x) = x\{\pi(\alpha-\gamma)/2 + \gamma \sin^{-1}(\theta/\beta) + \gamma((\beta^2 - \theta^2)^{\frac{1}{2}} - \beta)/\theta\} = cx,$$

where $c$ is a constant. Thus, $c_1(x)$ and $c_2(x)$ are given by the same equation. From equation (3.4), $h(s) = \gamma s$ and so the section shape which gives the required waterplane shape is similar to the one drawn in Figure 4.

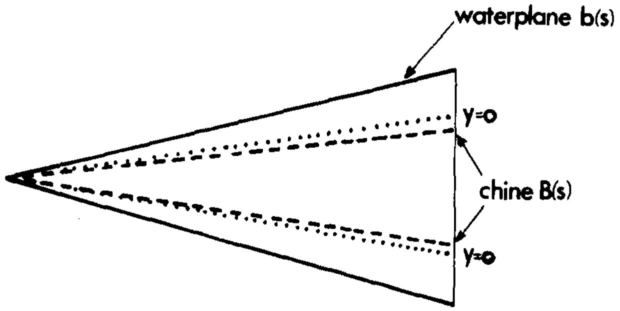


Figure 7. A test case with $\alpha = 0.1$, $\beta = 0.25$, $\gamma = 0.05$, $\theta = 0.125$ and $c = .375$.

This result was then used to test the numerical procedure. Values were chosen for $\alpha$, $\beta$, $\gamma$ and $\theta$, thus fixing $c(x)$. The inputs to the program were $\alpha$, $\gamma$ for the longitudinal hull slope, $\theta$ for the position of the chine and $c$ for the cross-section shape. The program output was $\beta$, which determines the position of the waterplane. Figure 7 gives a bird's eye view of the planing ship for particular values of $\alpha$, $\gamma$, $\theta$ and $c$, with the waterplane shape being the outer triangle of solid lines, the dashed lines showing the position of the chine and the dotted lines showing where the hull would have been wetted if the ship had been stationary ($y = 0$). The analytic and numerical results obtained for the position of the wetted area were so close together that on a graph they are coincident straight lines.

Although the above hull shape allows a comparison between analytic and numerical results, the hull form considered is not closely related to practical ship hulls. A more realistic shape is obtained by taking a hull given by

$$\eta(x,s) = \begin{cases} -0.15(s - 0.5s^2) + 0.4x, & x \leqslant B(s), \\ -0.15(s - 0.5s^2) + 0.8x - 0.04, & B(s) < x \leqslant b(s), \end{cases}$$

and the chine at a fixed offset, such that

$$B(s) = 0.1.$$

As the longitudinal hull slope $\eta_s(x, s)$ should not be zero on the hull, as separation will occur forward of such a point (see Casling [5]), a length of 0.5 was chosen for the ship.

The shape of the above hull and the numerical results for the waterplane shape are shown in Figure 8. For the given results, twenty points (equally spaced along the hull), were chosen and the solution iterated to an accuracy of three decimal places. The $b_i$'s calculated were used to recalculate equation (4.2) and the maximum absolute value of $F(b_i; b)$ was $6.8 \times 10^{-4}$, which occurred as the waterplane crossed the chine line.

In Figure 8, the waterplane $b(s)$ is shown, with the dashed line being the chine and the dotted line the position of the waterplane if the ship was stationary. On the cross-section, the crosses indicate the position of the edge of the waterplane and the dots its position if the ship was stationary. The side view results show that there is a rapid increase in the amount by which

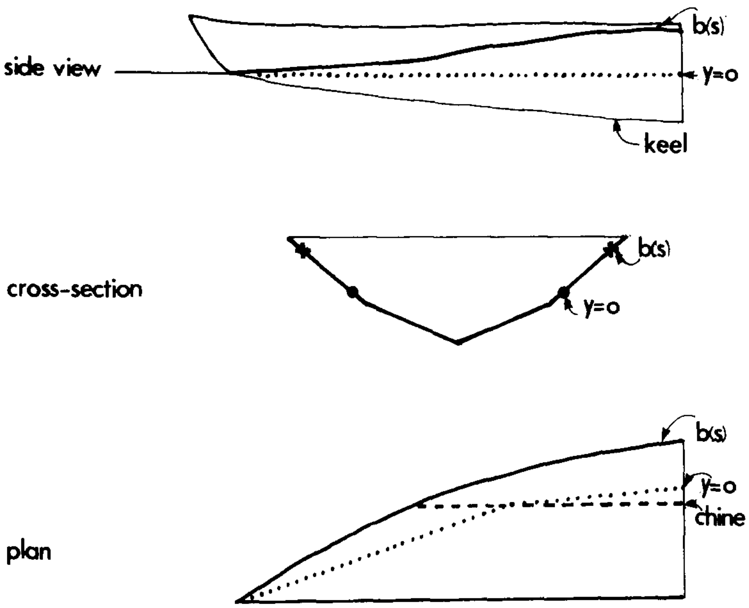


Figure 8. Waterplane shape for the second hull.

the free surface is displaced when the waterplane crosses the chine line. This could be expected as there is a rapid change in the hull slope in the x-direction at that position.

The above results show that the numerical procedure described in Section 4 works satisfactorily. On a CDC Cyber 173, a complete run giving three decimal place accuracy and estimating the error in the solution to equation (4.2) took 32 seconds. This means that the technique is sufficiently quick to be a practical tool.

## 6. The effect of a vertical chine

So far, we have assumed that the hull is wetted above the level of the chine and that the free-surface elevation is continuous across $x = b(s)$. However, it is more interesting, from a practical point of view, to be able to determine for a given hull shape whether or not the free surface actually rises above the level of the chine. One way of doing this is to consider a new hull which has no chine, but whose shape, both below and above the level of the chine on the original hull, is the same as the shape of the original hull below the level of its chine. That is, if the original hull is defined by

$$\eta(x,s) = \begin{cases} \int_0^s d\sigma\, \eta_\sigma^1(x,\sigma) + c_1(x), & x \leqslant B(s), \\ \\ \int_0^s d\sigma\, \eta_\sigma^2(x,\sigma) + c_2(x), & x > B(s), \end{cases}$$

then the new hull to be considered is given by

$$\eta(x,s) = \int_0^s d\sigma\, \eta_\sigma^1(x,\sigma) + c_1(x), \quad x \leqslant b(s).$$

From the results presented in Section 2, the function $x = b(s)$, which describes the shape of the waterplane of this hull, is unique. Therefore, whether or not the original hull is wetted above the chine depends on the position of the chine relative to the shape of the wetted area of the new hull. The problem divides itself into two cases.

Firstly if the quantity $B(s)$, which determines the position of the chine on the original hull, is always greater in value than the quantity $b(s)$, then, clearly, the free surface will not reach the chine. That is, if $B(s) \geqslant b(s)$ for all $s$, the original hull would not have been wetted above the chine (see Figure 9) and the shape of the waterplane is described by the function $x = b(s)$. In effect, the chine is irrelevant.

Secondly, if the curve $x = B(s)$ lies inside the wetted region of the new hull, that is, if $B(s) < b(s)$ for all $s$, then the original hull would have been wetted above the chine (see Figure 10). In this case, the waterplane shape is determined using the results derived earlier in this paper. It is, of course, possible to have a combination of these two cases.

There is a further possibility, which has not, as yet, been discussed – a vertical chine. By this, we mean that the hull has vertical sides along the curve $x = B(s)$ and so $\eta_x(s,B(s))$ is

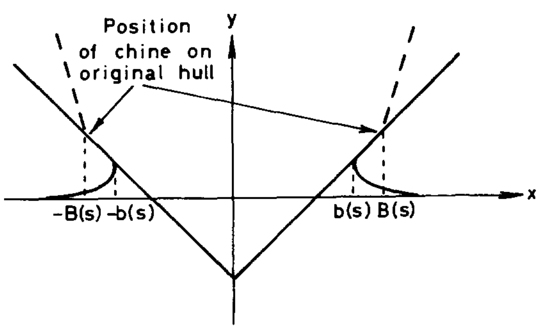


Figure 9. Hull not wetted above chine.

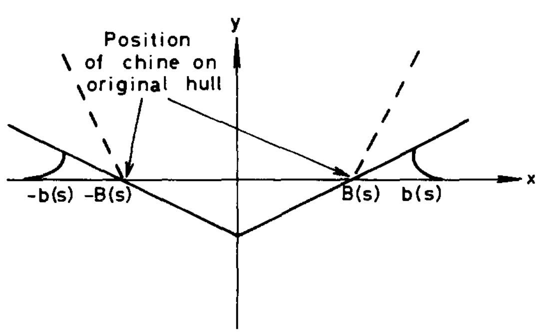


Figure 10. Hull wetted above chine.

infinite, and there is no function $c_2(x)$. Such a chine may be obtained by letting $h(s)$, the jump of $O(\alpha)$ in $\eta(x, s)$ at $x = B(s)$ (see equation (3.2)), tend to infinity. If $h(s) \to \infty$, then, from equation (3.4), $\eta(B(s)^+, s)$ must also become infinite. By considering the limit as $x \to B(s)^+$ in the expression for $\eta(x, s)$, given in equation (2.6), we see that $\eta(B(s)^+, s)$ will tend to infinity only if $b(s) \to B(s)^+$. Thus, as the size of the jump in the hull along the chine line increases, the amount of hull wetted past the chine decreases until, in the limiting case of a vertical chine, the function defining the waterplane is the same as the function describing the position of the chine. The result has been confirmed numerically.

This suggests that the best way of obtaining a required waterplane shape is to put a vertical chine along the curve which describes that shape, in the same way that a transom stern may be used to fix the wetted length of the hull. This may mean, however, that the linearised free-surface elevation is no longer finite along the leading edges of the hull.

If the vertical chine occurs along a fixed offset, $x = B$, starting at station $s = s_c$, then $b(s) = B$ when $s_c < s \leqslant L$ and so $b(s)$ has been fixed for these stations. When $0 \leqslant s \leqslant s_c$, $b(s)$ is to be determined as before. From equation (2.5), the free-surface elevation outside the hull is given by

$$\eta(x,s) = \begin{cases} -\displaystyle\int_0^s d\sigma\,(x^2 - b^2(\sigma))^{-\frac{1}{2}} H_{b(\sigma)}\eta_\sigma(x,\sigma)\,(b^2(\sigma) - x^2)^{\frac{1}{2}}, & 0 \leqslant s \leqslant s_c, \\[2ex] \displaystyle\int_0^{s_c} d\sigma\,(x^2 - b^2(\sigma))^{-\frac{1}{2}} H_{b(\sigma)}\eta_\sigma(x,\sigma)\,(b^2(\sigma) - x^2)^{\frac{1}{2}} & \\[2ex] -\dfrac{1}{\pi}(x^2 - B^2)^{-\frac{1}{2}} \displaystyle\int_{-B}^{B} \frac{d\xi}{x-\xi}\,(B^2 - \xi^2)^{\frac{1}{2}}(\eta^*(\xi,s) - \eta^*(\xi,s_c)), & s_c < s \leqslant L, \end{cases}$$

where

$$\eta^*(x,s) = \int ds\,\eta_s(x,s).$$

Taking the limit as $x \to B^+$ of this equation for $s_c < s \leqslant L$,

$$\eta(x,s) \to -\int_0^{s_c} d\sigma\,(B^2 - b^2(\sigma))^{-\frac{1}{2}}[H_{b(\sigma)}\eta_\sigma(x,\sigma)\,(b^2(\sigma) - x^2)^{\frac{1}{2}}]_{x=B}$$

$$-\frac{1}{\pi}(x^2 - B^2)^{-\frac{1}{2}} \int_{-B}^{B} d\xi (B+\xi)^{\frac{1}{2}}/(B-\xi)^{\frac{1}{2}}(\eta^*(\xi,s) - \eta^*(\xi,s_c)). \tag{6.1}$$

That is, $\eta(x, s) \to \infty$, as $x \to B^+$, when $s_c < s \leqslant L$ and the free-surface elevation is no longer finite along the side of the hull. So, the shape of the waterplane past $s = s_c$ has been fixed, but only at the expense of continuity in $\eta(x, s)$ across $x = b(s)$.

In particular, if $\eta_s(x, s)$ is independent of both $x$ and $s$, say $\eta_s(x, s) = -\alpha$, then

$$\eta(x,s) = \eta(x,s_c) - \alpha(s - s_c) + \alpha x(s - s_c)/(x^2 - B^2)^{\frac{1}{2}}.$$

This expression is identical to the equation obtained for the free-surface elevation outside the hull for $s > L$ in Casling [5] with $s_c$ substituted for $L$. Thus, the same flow field is obtained if the portion of the hull for $s > s_c$ is removed.

If the vertical chine occurs along the curve $x = B(s)$, starting at station $s = s_c$, then $b(s) \equiv B(s)$ when $s_c < s \leqslant L$. So the waterplane shape has been fixed for this range of $s$, but still must be determined for $0 \leqslant s \leqslant s_c$. The free-surface elevation outside the hull is given by

$$\eta(x,s) = \begin{cases} -\displaystyle\int_0^s d\sigma\, (x^2 - b^2(\sigma))^{-\frac{1}{2}} H_{b(\sigma)} \eta_\sigma(x,\sigma)\, (b^2(\sigma) - x^2)^{\frac{1}{2}}, & 0 \leqslant s \leqslant s_c, \\[2ex] -\displaystyle\int_0^{s_c} d\sigma\, (x^2 - b^2(\sigma))^{-\frac{1}{2}} H_{b(\sigma)} \eta_\sigma(x,\sigma)\, (b^2(\sigma) - x^2)^{\frac{1}{2}} & \\[2ex] -\dfrac{1}{\pi} \displaystyle\int_{s_c}^s d\sigma\, (x^2 - B^2(\sigma))^{-\frac{1}{2}} \int_{-B(\sigma)}^{B(\sigma)} \frac{d\xi}{x - \xi}\, \eta_\sigma(\xi,\sigma)\, (B^2(\sigma) - \xi^2)^{\frac{1}{2}}, & \\ & s_c < s \leqslant L. \end{cases}$$

The question of interest is now 'Is the free-surface elevation still finite along $x = b(s)$?'. The above equation for $\eta(x, s)$ may be written

$$\eta(x,s) = -\int_0^s d\sigma\, (x^2 - b^2(\sigma))^{-\frac{1}{2}} H_{b(\sigma)} \eta_\sigma(x,\sigma)\, (b^2(\sigma) - x^2)^{\frac{1}{2}}, \quad 0 \leqslant s \leqslant L,$$

where $b(s)$ is to be determined for $0 \leqslant s \leqslant s_c$ and is fixed for $s_c < s \leqslant L$. In the limit as $x$ tends to $b(s)$ from above, this integral is finite for all values of $s$ and so $\eta(b(s)^+, s)$ is finite and equals $\eta(b(s)^-, s)$.

A different result is obtained in this case, because the offset of the chine varies with the station, $s$. Thus, the square-root singularity $(x^2 - B^2(\sigma))^{-\frac{1}{2}}$, which appeared outside the integral in equation (6.1) when $B(s) = B$, now remains in the integrand and is integrable. When the chine occurs along a fixed offset, that is, $B(s) = B$, the singularity is independent of $s$ and cannot be integrated out.

Thus, by putting a vertical chine along a varying offset which starts at the bow, a particular waterplane shape may be chosen, provided that the hull is wetted up to the line of the chine, and the free-surface elevation will be finite along this curve.

## 7. Acknowledgement

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